2010

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. a. Let R be a region on the $xy-$ plane defined by

$$x^2 + y^2 \leq 1,\ x \geq 0,\ y \leq 0.$$

Find the integral

$$\iint_R e^{(x^2+y^2)} xydxdy.$$

Use polar coordinates, r, ϕ. Then R is defined by $0 \leq r \leq 1$, $\frac{3\pi}{2}\phi \leq 2\pi$ and

$$\iint_R e^{(x^2+y^2)} xydxdy = \int_0^1 \int_{\frac{3\pi}{2}}^{2\pi} \exp\left(r^2\right) r^3 \cos(\phi) \sin(\phi)\, drd\phi$$

$$= \frac{1}{8}\left(\int_0^1 e^u udu\right)\left(\int_{3\pi}^{4\pi} \sin(\alpha)\, d\alpha\right)$$

$$= -\frac{1}{8}\left[ue^u - e^u\right]_0^1 \left[\cos(\alpha)\right]|_{3\pi}^{4\pi} = -\frac{1}{4}.$$

b. Let the surface S be the graph of the function $f(x, y) = 1 + x^2 + y^2$, where (x, y) satisfy

$$|x| + |y| \leq 1.$$

Find the surface integral

$$\iint_S \frac{e^{(x+y)}}{\sqrt{4z-3}}\, dS.$$

[Hint: Use the change of variables: $u = x + y$, $v = x - y$.]

If S is the graph of a function f over R, then

$$\iint_S g(x, y, z)\, dS = \iint_R g(x, y, f(x, y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy.$$

Also, in the coordinates u, v, the region R is defined by

$$-1 \leq u \leq 1, \quad -1 \leq v \leq 1.$$

At last, Jacobian of the above transformation, is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}.$$

Thus,

$$\iint_S \frac{\exp(x+y)}{\sqrt{4z-3}}\,dS$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} \frac{\exp(u)}{\sqrt{4z-3}}\sqrt{1+4x^2+4y^2}\,dudv = (e-e^{-1}).$$

2. a*. State the Divergence Theorem carefully.

Let D be a bounded domain in $\mathcal{R}^3$ surrounded by a smooth surface S. Let

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k},$$

be a smooth vector-field in D. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dxdydz.$$

b. Let D be a cylinder,

$$x^2+y^2 \le 1, \quad 0 \le z \le 1.$$

Let $\mathbf{F}$ be a vector field

$$\mathbf{F}(x,y,z) = a^2xy^2\mathbf{i} + a^2yx^2\mathbf{j} + z^2\mathbf{k},$$

where a is a real number. Find the flux of $\mathbf{F}$ through S, where S is the surface surrounding D.

We have

$$\nabla\cdot\mathbf{F} = a^2y^2 + a^2y^2 + 2z = a^2(x^2+y^2) + 2z.$$

By the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \left(a^2(x^2+y^2)+2z\right)\,dxdydz.$$

Using the cylindrical coordinates, ρ, z, ϕ, we get

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = a^2\int_0^1\int_0^{2\pi}\int_0^1 \rho^3 d\rho dz d\phi + 2\int_0^1\int_0^{2\pi}\int_0^1 \rho z d\rho dz d\phi$$

$$= (1+\frac{1}{2}a^2)\pi.$$

c. For the function $f(x,y) = x + e^{xy}$ find the equation of its tangent plane at the point $(0,1,1)$.

We have,

$$\nabla f(x,y) = (1 + ye^{xy})\mathbf{i} + xe^{xy}\mathbf{j}.$$

Thus, $\nabla f(0,1) = 2\mathbf{i}$ so that the tangent plane is

$$z = 2 + 2x + 0(y-1) = 2 + 2x.$$

d. For the function f from Part c., find a vector $\mathbf{u} \neq \mathbf{0}$ which is orthogonal to $\nabla f(0,1)$.

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$,

$$\nabla f(0,1) \cdot \mathbf{u} = 2u_1.$$

Thus $\mathbf{u} = \mathbf{j}$ is a possible choice.

3. a. State Stoke's Theorem carefully.

Given a curve C with the anti-clockwise direction and a capping surface S, so that C is the boundary of S, let $\mathbf{F}(x,y,z)$ be a smooth vector field defined on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

b. Verify Stoke's Theorem for the vector field

$$\mathbf{F}(x,y,z) = zy\mathbf{i} + z\mathbf{k}$$

and the surface S defined by

$$x^2 + y^2 + z^2 =, \quad z \geq \sqrt{3}.$$

i. The curve C is the circle $x^2 + y^2 = 1$ lying on the plane $z = \sqrt{3}$. Use parametrization

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sqrt{3}\mathbf{k}, \quad \mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}.$$

Thus,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\sqrt{3}\sin^2(t),$$

so that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -\sqrt{3}\int_0^{2\pi} \sin^2(t)\, dt = -\sqrt{3}\pi.$$

ii. On the other hand,

$$\nabla \times \mathbf{F} = y\mathbf{j} - z\mathbf{k},$$

and S is the graph of the function $f(x,y) = \sqrt{4 - x^2 - y^2}$ with the domain $R = \{(x,y) : x^2 + y^2 \le 1\}$. As

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}},$$

we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R \left(\frac{y^2}{\sqrt{4-x^2-y^2}} - z \right) dxdy$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^2 \sin^2(\phi)}{\sqrt{4-r^2}} - \sqrt{4-r^2} \right) r dr d\phi = \pi \int_0^1 \left(\frac{r^2 - 2(4-r^2)}{\sqrt{4-r^2}} \right) rdr$$

$$= \frac{\pi}{2} \int_3^4 \frac{4-3t}{t^{1/2}} dt = -\sqrt{3}\pi.$$

4. a. State Green's Theorem in the plane carefully.

Let C be a smooth closed curve in the $xy-$plane oriented in the anti-cloackwise direction. Let R be a bounded region surrounded by C. Suppose that

$$\mathbf{F}(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$$

be a smooth vector field on R. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

b. Prove Green's Theorem.

Hint: You may use Stoke's Theorem with $\mathbf{F}(x,y,z) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$.

Let S lie in the $xy-$plane, $z = 0$. Then $\mathbf{n} = \mathbf{k}$, while

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}$$

Using the Stoke's Theorem we obtain

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

c. Let the contour C consists of an interval, $-2 \le x \le 2$, lying on the $x-$axis, and a semicircle $x^2 + y^2 = 4$, $y \ge 0$. The contour C has an anti-clockwise direction. Let

$$\mathbf{F}(x,y) = (\sin(xy) + xy\cos(xy))\,\mathbf{i} + \left(x^2\cos(xy) + xy\right)\mathbf{j}.$$

Use Green's Theorem to calculate the circulation of $\mathbf{F}$ around C.

We have

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y.$$

Thus, in polar coordinates,

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int_0^2 \int_0^\pi r^2 \sin(\phi)\, dr d\phi = \frac{8}{3} \cdot 2 = \frac{16}{3}.$$

5. a. Let $D = \mathcal{R}^3 \setminus \{(x,y,z) : x = y = 0\}$. Let C be a smooth curve in D. Let $\mathbf{F}(x,y,z)$ be a smooth vector field in D such that

$$\nabla \times \mathbf{F} = \mathbf{0}, \quad \text{in } \mathbf{D}.$$

Does it guarantee that $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the initial, $\mathbf{X_0}$, and terminal, $\mathbf{X_1}$, points of C? Expain your answer.
(Note that $\mathcal{R}^3 \setminus \{(x,y,z) : x = y = 0\}$ is the whole $\mathcal{R}^3$ without the $z-$axis).

Since D is not simply connected, the fact that $\nabla \times \mathbf{F} = \mathbf{0}$ does not guarantee that $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on $\mathbf{X_0}$ and $\mathbf{X_1}$.

b. ı) Let

$$\mathbf{F} = ye^{xy}\ln(1+z^2)\mathbf{i} + \left(xe^{xy}\ln(1+z^2) + 1\right)\mathbf{j} + \frac{2ze^{xy}}{1+z^2}\mathbf{k}.$$

Show that $\nabla \times \mathbf{F} = \mathbf{0}$ and find a potential function for $\mathbf{F}$.

Consider

$$\mathbf{E} = \nabla \times \mathbf{F} = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k}.$$

We have

$$E_1 = \frac{\partial}{\partial y}\left(\frac{2ze^{xy}}{1+z^2}\right) - \frac{\partial}{\partial z}\left(xe^{xy}\ln(1+z^2) + 1\right) = 0;$$

$$E_2 = \frac{\partial}{\partial z}\left(ye^{xy}\ln(1+z^2)\right) - \frac{\partial}{\partial x}\left(\frac{2ze^{xy}}{1+z^2}\right) = 0;$$

$$E_3 = \frac{\partial}{\partial x}\left(\frac{2ze^{xy}}{1+z^2}+1\right) - \frac{\partial}{\partial y}\left(ye^{xy}\ln(1+z^2)\right) = 0.$$

As $\mathcal{R}^3$ is simply connected, there exists function $f(x,y,z)$ such that

$$\mathbf{F} = \nabla f.$$

Then

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{2ze^{xy}}{1+z^2},$$

so that

$$f(x,y,z) = e^{xy}\ln(1+z^2) + g(x,y).$$

Now,

$$\frac{\partial f}{\partial x}(x,y,z) = ye^{xy}\ln(1+z^2),$$

so that

$$f(x,y,z) = e^{xy}\ln(1+z^2) + g(y).$$

At last,

$$\frac{\partial f}{\partial y}(x,y,z) = xe^{xy}\ln(1+z^2) + 1,$$

so that

$$f(x,y,z) = e^{xy}\ln(1+z^2) + y + C.$$

c. Let

$$\mathbf{G} = \left(ye^{xy}\ln(1+z^2) + x\right)\mathbf{i} + \left(xe^{xy}\ln(1+z^2)+1\right)\mathbf{j} + \frac{2ze^{xy}}{1+z^2}\mathbf{k}.$$

Let C be a curve of the form $x = t$, $y = 0$, $z = t(1-t)$ with $0 \le t \le 1$ which connects the point $\mathbf{X}_0 = \mathbf{0}$ with the point $\mathbf{X}_1 = (1,0,0)$.
Find

$$\oint_C \mathbf{G}\cdot d\mathbf{r}.$$

[Hint: Use Part (b).]

As $\mathbf{G} = \mathbf{F} + x\mathbf{i}$, due to Part (b),

$$\int_C \mathbf{G}\cdot d\mathbf{r} = \int_C x\mathbf{i}\cdot d\mathbf{r} + f(\mathbf{X}_1) - f(\mathbf{X}_0) = \int_C x\mathbf{i}\cdot d\mathbf{r} = \int_0^1 t dt = \frac{1}{2}.$$

6. a. State Fourier's Theorem carefully.

Let $f(x)$ be a $2\pi-$periodic piecewise continuously differentiable function. Then $f(x)$ can be represented by trigonometric Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right),$$

where a_0, a_n, b_n, $n = 1, 2, \ldots$ are its Fourier coefficients. The Fourier series at x converges to $f(x)$, if x is a point of continuity of f. If $f(x)$ has a jump in x, then the Fourier series at x converges to $\frac{1}{2}(f(x-0) + f(x+0))$, where

$$f(x \pm 0) = \lim_{\epsilon \to +0} f(x \pm \epsilon).$$

b. Find the Fourier coefficients of the function $f(x)$ which is equal to

- 0 for $-\pi \leq x \leq 0$;
- x for $0 \leq x \leq \pi$;
- continued 2π periodically from $(-\pi, \pi)$ to the whole $\mathcal{R}$.

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} x dx = \frac{\pi}{4};$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos(nx) dx = 0, \text{ if } n = 2k,$$

$$a_n = \frac{-2}{\pi(2k-1)^2}, \text{ if } n = 2k-1;$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin(nx) dx = \frac{(-1)^{(n+1)}}{n}.$$

END OF PAPER